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A STUDY OF GRAPH CLOSED SUBSEMIGROUPS OF A FULL TRANSFORMATION SEMIGROUP

BY

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ABSTRACT. Let T_X be the full transformation semigroup on the set X and let S be a subsemigroup of T_X. We may associate with S a digraph $g(S)$ with X as set of vertices as follows: $i \longrightarrow j \in g(S)$ iff there exists $\alpha \in S$ such that $\alpha(i) = j$. Conversely, for a digraph G having certain properties we may assign a semigroup structure, $S(G)$, to the underlying set of G. We are thus able to establish a "Galois correspondence" between the subsemigroups of T_X and a particular class of digraphs on X. In general, S is a proper subsemigroup of $S \cdot g(S)$.

1. **Introduction.** In this article, we shall focus our attention on graph closed subsemigroups of T_X, i.e., subsemigroups S for which $S \cdot g(S) = S$. As might be expected, there is a close relationship between this class of semigroups and the "geometry" of the associated graphs.

The class of graph closed nil subsemigroups is particularly amenable to treatment via the "geometric" properties of the corresponding graphs. This is due, in part, to the fact that the graph of a nil subsemigroup of T_X may be interpreted as a partial order on X. Graph closed nil subsemigroups are therefore the prime objects of our investigation. In particular, we determine the maximal nil subsemigroups of T_X and the maximal nilpotent semigroups of given nilpotency index and graph type. For finite X, we compute the order of closed nil subsemigroups of given nilpotency index and graph type. Furthermore, still for finite X, the number of isomorphism classes of maximal nil subsemigroups is determined and we show that every graph closed subsemigroup of a maximal nil semigroup N is a meta-ideal of N.

Throughout this paper $E(S)$ denotes the set of idempotents of S, N denotes the natural numbers, the number of elements in a set X is denoted by $|X|$ and mappings are composed from right to left. For a finite set X, say $X = \{1, 2, \ldots, n\}$, we use the notation of [1, p. 54] for elements of T_X, namely $\alpha = (i_1 i_2 \cdots i_n)$ is the function $\alpha(j) = i_j$ for all $j \in X$. This notation is particularly useful for com-

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puting products, since $(i_1 i_2 \cdots i_n)(j_1 j_2 \cdots j_n) = (i_{j_1} i_{j_2} \cdots i_{j_n})$.

2. **Algebraic graphs.** Let X be a set and G a directed graph on X. For any $i, j \in X$, we shall write $i \longrightarrow j \in G$ to mean that the directed edge (i, j) belongs to the graph G. A directed graph G is said to be transitive if for all i, j and k in X, $i \longrightarrow j \in G$ and $j \longrightarrow k \in G$ implies that $i \longrightarrow k \in G$.

2.1. DEFINITION. A transitive graph G on a set X is said to be algebraic on X if for all $i \in X$ there exists $j \in X$ such that $i \longrightarrow j \in G$.

Let S_X denote the set of all subsemigroups of the full transformation semigroup T_X for a set X, and let G_X be the set of all algebraic graphs on X. We introduce a function g from S_X to G_X and a function S from G_X to S_X such that $g \cdot S$ is the identity on G_X and $S \cdot g$ is a closure operator on S_X.

2.2. DEFINITION. Let $S \in S_X$. Construct a directed graph $g(S)$ on X as follows: for each $i, j \in X$, $i \longrightarrow j \in g(S)$ iff there exists $\alpha \in S$ such that $\alpha(i) = j$.

It is clear that for each semigroup $S \in S_X$, $g(S) \in G_X$ and so we have a function $g\colon S_X \longrightarrow G_X$.

2.3. DEFINITION. Let $G \in G_X$. Define $S(G) = \{\alpha \in T_X \mid \text{for each } i \in X, \alpha(i) = j \text{ implies that } i \longrightarrow j \in G\}$.

Since G is algebraic, it follows that $S(G)$ is a semigroup, i.e. $S(G) \in S_X$.

The following lemma is an immediate consequence of the above definitions.

2.3. LEMMA. (i) *If $S_1, S_2 \in S_X$ and $S_1 \subset S_2$, then $g(S_1) \subset g(S_2)$.*
(ii) *If $G_1, G_2 \in G_X$ and $G_1 \subset G_2$, then $S(G_1) \subset S(G_2)$.*

2.4. THEOREM. *$g \cdot S$ is the identity on G_X and $S \cdot g$ is a closure operator on S_X.*

PROOF. First we prove that $g \cdot S$ is the identity on G_X. Let $G \in G_X$ and suppose that $i \longrightarrow j \in G$. Define $\alpha \in T_X$ by $\alpha(i) = j$, while for each $k \in X \backslash \{i\}$, $\alpha(k) = l$ where l is any element of X such that $k \longrightarrow l \in G$. At least one such l must exist for each k, since G is algebraic. Thus $\alpha \in S(G)$, and so $i \longrightarrow j \in g \cdot S(G)$. Thus $G \subset g \cdot S(G)$. Conversely, if $i \longrightarrow j \in g \cdot S(G)$, there exists $\alpha \in S(G)$ such that $\alpha(i) = j$. This implies that $i \longrightarrow j \in G$ and so $g \cdot S(G) \subset G$. Thus $G = g \cdot S(G)$.

It is easily seen that $S \cdot g$ is a closure operator on S_X, i.e. that $S \cdot g$ has the following properties: (i) for each $S \in S_X$, $S \subset S \cdot g(S)$, (ii) if $S, T \in S_X$ and $S \subset T$, then $S \cdot g(S) \subset S \cdot g(T)$, and (iii) $S \cdot g$ is idempotent.

2.5. DEFINITION. For each $S \in S_X$, let $\overline{S} = S \cdot g(S)$.

2.6. DEFINITION. For each directed graph G on a set X, we define a directed graph $I(G)$, called the idealized graph of G (for reasons which will be apparent in Lemma 2.10), as follows: let $i, j \in X$. Then $i \longrightarrow j \in \mathcal{I}(G)$ iff for all $k \in X$:

(i) $j \longrightarrow k \in G$ implies that $i \longrightarrow k \in G$, and

(ii) $k \longrightarrow i \in G$ implies that $k \longrightarrow j \in G$.

Observe that the idealized graph of any algebraic graph contains the directed edges $i \longrightarrow i \; \forall i \in X$.

2.7. THEOREM *If $G \in G_X$, then $\mathcal{I}(G) \in G_X$ and $G \subset \mathcal{I}(G)$.*

PROOF. Suppose $G \in G_X$. To show that $\mathcal{I}(G) \in G_X$, it is sufficient to show that $G \subset \mathcal{I}(G)$ and $\mathcal{I}(G)$ is transitive. Let $i \longrightarrow j \in G$. Then for all $k \in X$, if $j \longrightarrow k \in G$, then $i \longrightarrow k \in G$, while if $k \longrightarrow i \in G$, then $k \longrightarrow j \in G$. Thus $i \longrightarrow j \in \mathcal{I}(G)$; and so $G \subset \mathcal{I}(G)$. Now, to prove that $\mathcal{I}(G)$ is transitive, let $i \longrightarrow j \in \mathcal{I}(G), j \longrightarrow k \in \mathcal{I}(G)$. We must show that $i \longrightarrow k \in \mathcal{I}(G)$. Let $k \longrightarrow l \in G$. Then $j \longrightarrow l \in G$ and so $i \longrightarrow l \in G$. On the other hand, if $l \longrightarrow i \in G$, then $l \longrightarrow j \in G$ and thus $l \longrightarrow k \in G$. Therefore $i \longrightarrow k \in \mathcal{I}(G)$.

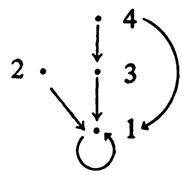
Let us return now to a consideration of the closure operator on S_X. The natural course to take, when one has a closure operator, is to single out the "closed" objects for investigation.

2.8. DEFINITION. A semigroup $S \in S_X$ is said to be graph closed if $S = \overline{S}$.

It is clear that not all elements of S_X are graph closed. For example, the only subgroups of T_X which are graph closed are the trivial groups. Thus graph closed semigroups are far removed from groups. An interesting class of graph closed semigroups, as we shall see, are the graph closed nil subsemigroups of T_X. We begin with an investigation of the properties of graph closed semigroups in general. The first observation to be made is that a graph closed semigroup is completely described by its graph. A convenient presentation of a graph closed semigroup on a finite set can be obtained by a table in the following way: let $X = \{1, 2, \ldots, n\}$, and let $S \in S_X$ be graph closed. Make a table with n columns, one for each element of X, and in the column corresponding to $i \in X$, write down all elements $j \in X$ such that $i \longrightarrow j \in g(S)$. Then an element α of S is obtained by choosing one element from each column, say j_i is chosen from the ith column for each $i \in X$, and setting $\alpha = (j_1 j_2 \cdots j_n)$. Moreover, each element of S is obtained in this way, and distinct choices obviously give different elements of S. For each $i \in X$, let n_i be the number of elements in the ith column, i.e. n_i is the number of elements $j \in X$ such that there exists $\alpha \in S$ with $\alpha(i) = j$.

2.9. THEOREM (COUNTING LEMMA). *Let X be a finite set and $S \in S_X$ be graph closed. Then $|S| = \Pi_{i \in X} n_i$.*

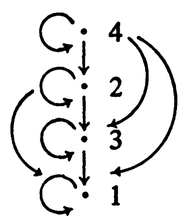
For example, let $X = \{1, 2, 3, 4\}$ and G be the algebraic graph



The table for $\mathcal{S}(G)$ is

1	2	3	4
1	1	1	1
			3

and $\mathcal{S}(G) = \{(1111), (1113)\}$. In addition, consider the idealized graph $\mathcal{I}(G)$ of G:



with table

1	2	3	4
1	1	1	1
	2	3	2
	3		3
			4

The order of $\mathcal{S} \cdot \mathcal{I}(G)$ is (3) (2) (4) = 24. It is easily seen in this example that $\mathcal{S}(G)$ is an ideal in $\mathcal{S} \cdot \mathcal{I}(G)$. In fact, $\mathcal{S} \cdot \mathcal{I}(G)$ is the idealizer of $\mathcal{S}(G)$ and the following lemma asserts that this is true in general.

2.10. Lemma. *Let $S \in \mathcal{S}_X$ be graph closed. Then the idealizer $\mathcal{I}(S)$ of S in $\mathcal{T}_X$ is graph closed and $g \cdot \mathcal{I}(S) = \mathcal{I} \cdot g(S)$.*

Proof. We prove that $\mathcal{I}(S) = S \cdot \mathcal{I} \cdot g(S)$ whence $\mathcal{I}(S)$ is graph closed and by (2.4), $g \cdot \mathcal{I}(S) = g \cdot S \cdot \mathcal{I} \cdot g(S) = \mathcal{I} \cdot g(S)$. Let $G = \mathcal{I} \cdot g(S)$. We begin by showing that $\mathcal{S}(G) \subset \mathcal{I}(S)$. Let $\alpha \in \mathcal{S}(G)$ and $\beta \in S$. Then for each $i \in X$, $i \longrightarrow \beta(i) \in g(S)$ and $\beta(i) \longrightarrow \alpha\beta(i) \in G$. Thus $i \longrightarrow \alpha\beta(i) \in g(S)$.

Since S is graph closed, $\alpha\beta \in S$. Similarly, $\beta\alpha \in S$. Thus $\alpha \in \mathcal{I}(S)$ and so $S(G) \subset \mathcal{I}(S)$.

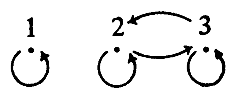
Conversely we prove that $\mathcal{I}(S) \subset S(G)$. Suppose this is not so, and choose $\beta \in \mathcal{I}(S)\setminus S(G)$. Then there exist $i, j \in X$ such that $i \longrightarrow j \notin G$ but $\beta(i) = j$. It follows that for some $k \in X$, either

(i) $k \longrightarrow i \in g(S)$ but $k \longrightarrow j \notin g(S)$, or else

(ii) $j \longrightarrow k \in g(S)$ but $i \longrightarrow k \notin g(S)$.

If (i) is the case, then there exists $\alpha \in S$ such that $\alpha(k) = i$. However, $\beta\alpha \in S$ and $\beta\alpha(k) = \beta(i) = j$ and so $k \longrightarrow j \in g(S)$, a contradiction. If (ii) holds, then there exists $\alpha \in S$ such that $\alpha(j) = k$. But $\alpha\beta \in S$ and $\alpha\beta(i) = \alpha(j) = k$, whence $i \longrightarrow k \in g(S)$, again a contradiction. Thus $\mathcal{I}(S)\setminus S(G) = \varnothing$.

This lemma implies that for each graph closed semigroup $S_0 \in \mathcal{S}_X$, a sequence of graph closed semigroups $S_i \in \mathcal{S}_X$ can be found such that $S_0 \subset S_1 \subset \cdots$ and $S_{i-1} \,\Delta\, S_i$ for each $i \geqslant 0$. However, even for finite X, it is not possible in general to obtain such a chain with $S_i = \mathcal{T}_X$ for some i, i.e. a graph closed subsemigroup of $\mathcal{T}_X$ is not necessarily a meta-ideal of $\mathcal{T}_X$. For this would be equivalent to the following: for all graph closed $S \in \mathcal{S}_X$, $\mathcal{I} \cdot g(S) = g(S)$ implies that $g(S)$ is the graph of $\mathcal{T}_X$, namely the complete graph on X, and if S is graph closed, then $S = \mathcal{T}_X$. But consider the following example. Let $X = \{1, 2, 3\}$ and G be the algebraic graph on X given by



Clearly, $\mathcal{I}(G) = G$ and G is not the complete graph on X. In the last section, it will be shown that for finite X, any graph closed subsemigroup of a maximal nil semigroup in $\mathcal{T}_X$ is a meta-ideal of the maximal nil semigroup.

The final observation of this section is the not unexpected result that two isomorphic graph closed semigroups are embedded in $\mathcal{T}_X$ in the same way iff their graphs are isomorphic.

2.11. THEOREM. *Let S_1 and S_2 be graph closed subsemigroups of $\mathcal{T}_X$. Then $g(S_1) \simeq g(S_2)$ iff there exists an automorphism h of $\mathcal{T}_X$ such that h induces an isomorphism between S_1 and S_2.*

PROOF. Let $g(S_1) \simeq g(S_2)$ be an isomorphism α. Then α can be considered as an element of $\mathrm{Sym}(X) \subset \mathcal{T}_X$. Let $c_\alpha : \mathcal{T}_X \longrightarrow \mathcal{T}_X$ denote conjugation by α, i.e. $c_\alpha(\gamma) = \alpha\gamma\alpha^{-1}$ for all $\gamma \in \mathcal{T}_X$. Then $c_\alpha \in \mathrm{Aut}(\mathcal{T}_X)$. We prove that $c_\alpha | S_1$ is an isomorphism from S_1 onto S_2. It is sufficient to prove that $c_\alpha(S_1) \subset S_2$ since by symmetry of argument, $c_\alpha^{-1}(S_2) \subset S_1$ and so c_α is surjective, whence it is bijective. Let $\gamma \in S_1$. Then $c_\alpha(\gamma) = \alpha\gamma\alpha^{-1}$. We must show that for all $i \in X$, $i \longrightarrow \alpha\gamma\alpha^{-1}(i) \in g(S_2)$. Let $j = \alpha^{-1}(i)$, and $k = \gamma(j)$. Then $j \longrightarrow k \in g(S_1)$ and so $\alpha(j) \longrightarrow \alpha(k) \in g(S_2)$. But $\alpha(j) = i$, $\alpha(k) = \alpha\gamma(j) = \alpha\gamma\alpha^{-1}(i)$ and so $i \longrightarrow \alpha\gamma\alpha^{-1}(i) \in g(S_2)$. Thus $c_\alpha | S_1$ is an isomorphism from S_1 onto S_2.

Conversely, suppose there exists an automorphism h of T_X whose restriction to S_1 is an isomorphism from S_1 onto S_2. Since every automorphsim of T_X is inner [2], there exists $\alpha \in \mathrm{Sym}(X)$ such that $h = c_\alpha$. Now α is a bijection on X. We prove that α preserves direction. Let $i \longrightarrow j \in g(S_1)$. Then there exists $\beta \in S_1$ such that $\beta(i) = j$. Let $c_\alpha(\beta) = \gamma \in S_2$. Then $\beta = \alpha^{-1}\gamma\alpha$ and so $\alpha^{-1}\gamma\alpha(i) = j$, whence $\gamma(\alpha(i)) = \alpha(j)$ and so $\alpha(i) \longrightarrow \alpha(j) \in g(S_2)$. By symmetry, $(c_\alpha)^{-1} = c_\alpha^{-1}$ also preserves direction. Thus $g(S_1) \simeq g(S_2)$.

3. **Nil semigroups of T_X.** In this section we show that the set of graph closed nil subsemigroups of T_X is a very tractable class of graph closed elements of S_X to study. The principal observation to be made is that the graph of a nil element of S_X "is" a partial order on X having certain properties. Many of the observations to be made about nil graph closed elements of S_X are intuitively clear from a consideration of the graph of the semigroup.

3.1. LEMMA. *Let $S \in S_X$ be nil with zero ξ. Then the set of fixed points of each element of S is $\xi(X)$.*

PROOF. Let $\alpha \in X$. For each $i \in \xi(X)$, $i = \xi(j)$ for some $j \in X$ and so $\alpha(i) = \alpha\xi(j) = \xi(j) = i$. Thus α fixes i. Now suppose that for some $j \in X$, $\alpha(j) = j$. Then for each positive integer n, $\alpha^n(j) = j$. Since S is nil, $\alpha^n = \xi$ for some n. Thus $\xi(j) = j$ and so $j \in \xi(X)$.

3.2. DEFINITION. For a nil semigroup $S \in S_X$ with zero ξ, let $\mathrm{Fix}(S) = \xi(X)$. In particular, $\mathrm{Fix}(\{\xi\})$ is denoted by $\mathrm{Fix}(\xi)$.

We now establish that the relation on X determined by $g(S)$ for nil S is a partial order with set of minimal elements $\mathrm{Fix}(S)$ and having the property that every element of $X \backslash \mathrm{Fix}(S)$ is greater than exactly one element of $\mathrm{Fix}(S)$.

3.3. DEFINITION. Let $S \in S_X$ be nil. Then for $i, j \in X$, let $i \geqslant_S j$ iff $i = j$ or there exists $\alpha \in S$ such that $\alpha(i) = j$.

This is clearly a reflexive, transitive relation on X. Suppose that $i \geqslant_S j$ and $j \geqslant_S i$. If $i \neq j$, then there exist $\alpha, \beta \in S$ such that $\alpha(i) = j$ and $\beta(j) = i$ whence $\beta\alpha(i) = i$ and so $i \in \mathrm{Fix}(S)$. But then $i = j$, a contradiction. Thus $i = j$ and $\geqslant_S$ is antisymmetric, hence a partial order on X said to be induced by S. We remark that if $S, T \in S_X$ are nil and $S \subset T$, then $i \leqslant_S j$ implies $i \leqslant_T j$.

3.4. LEMMA. *The partial order on X induced by a nil semigroup $S \subset T_X$*

has minimal elements, and every element which is not minimal is greater than exactly one minimal element. The set of all minimal elements is $\mathrm{Fix}(S)$.

PROOF. Let $i \in \mathrm{Fix}(S)$. Then $\alpha(i) = i$ for all $\alpha \in S$ and thus i is minimal for $\geqslant_S$. Suppose conversely that $i \in X$ is minimal. Then $\xi(i) = i \in \mathrm{Fix}(S)$, where ξ is the zero of S.

It is clear that for every $i \in X$, $\xi(i)$ is the unique element of $\mathrm{Fix}(S)$ which is less than or equal to i.

Let X be a set. For the remainder of this section, S shall denote a nil element of S_X with zero ξ.

3.5. DEFINITION. For each $i \in X$, let $X_i = X_i(S) = \{ j \in X \mid j \geqslant_S i\}$.

The preceding lemma implies that the set $\{X_i \mid i \in \mathrm{Fix}(S)\}$ is a partition of X.

3.6. THEOREM. *For each* $i \in \mathrm{Fix}(S)$, *there exists a nil semigroup* $S_i \subset T_{X_i}$ *such that*

(i) $|\mathrm{Fix}(S_i)| = 1$,

(ii) *each* S_i *is a retract of* S *in such a way that* $S_i S_j = \{\xi\}$ *for all* $i, j \in \mathrm{Fix}(S)$, $i \neq j$,

(iii) S *is a subdirect product of the* S_i. *If* S *is graph closed, then* $S \simeq \Pi_{i \in \mathrm{Fix}(S)} S_i$ *and each* S_i *is graph closed.*

PROOF. First, observe that (3.4) implies that for each $i \in \mathrm{Fix}(S)$ and each $\alpha \in S$, $\alpha | X_i \in T_{X_i}$. Let $S_i = \{\alpha | X_i \mid \alpha \in S\}$. It is obvious that S_i is a nil subsemigroup of T_{X_i} with zero $\xi | X_i$, and $\xi(X_i) = \{i\}$. Thus $|\mathrm{Fix}(S_i)| = 1$. Define for each $i \in \mathrm{Fix}(S)$ a function $I_i : S_i \longrightarrow S$ as follows: for $\alpha \in S_i$, let $I_i(\alpha) | X_i = \alpha$ and $I_i(\alpha) | X \backslash X_i = \xi | X \backslash X_i$. It is clear that I_i is an injective homomorphism. Now let $\Pi_i : S \longrightarrow S_i$ be the restriction homomorphism. We have $\Pi_i \cdot I_i = \mathrm{id}_{S_i}$ and so S_i is a retract of S. By the definition of I_i, (the images of) S_i and S_j satisfy $S_i S_j = \{\xi\}$ for $i, j \in \mathrm{Fix}(S)$, $i \neq j$. From the definition of the S_i, statement (iii) is clear.

REMARK. If $|X_i| \leqslant 2$ for some $i \in \mathrm{Fix}(S)$, then $|S_i| = 1$.

3.7. DEFINITION. Let $G \in G_X$. If there exists a subset $F(G)$ of X such that

(i) for all $i \in X$, $i \longrightarrow i \in G$ iff $i \in F(G)$, and

(ii) for all $j \in X$, there exists a unique $i \in F(G)$ such that $j \longrightarrow i \in G$,

then G is said to be a nil graph. For each $i \in X$, let i_F be the unique element of $F(G)$ whose existence is asserted by (ii).

It is readily apparent that if $S \in S_X$ is nil, then $g(S)$ is nil, while if $G \in G_X$ is nil, then $S(G)$ is nil. We denote by $\leqslant_G$ the partial order on X induced by

$S(G)$. For $i, j \in X$ with $i \leqslant_G j$, the closed interval with end points i and j shall be denoted by $[i, j]_G$. If $[i, j]_G$ is a chain with $n + 1$ elements, it is said to be of length n. Finally, a nil graph for which every chain is of length less than or equal to k, and some chain is of length k, is called a k-nilpotent graph. For a nilpotent graph G on a set X, we can define a function from X into N, called the height function on X corresponding to G.

3.8. DEFINITION. Let $G \in G_X$ be k-nilpotent and consider the partial order $\leqslant_G$ induced on X by G. The function $h_G: X \longrightarrow \{0, 1, 2, \ldots, k\}$ defined by setting $h_G(i)$ equal to the length of the longest chain in $[i_F, i]_G$ is called the height function on X induced by G.

A k-nilpotent semigroup is a nilpotent semigroup of nilpotency index k. If $S \in S_X$ is k-nilpotent, then $g(S)$ is k-nilpotent. For nilpotent S, let $h_S = h_{g(S)}$.

3.9. LEMMA. (i) *h_G is surjective.*

(ii) *$h_G(i) = 0$ iff $i \in F(G)$.*

(iii) *If $i >_G j$, then $h_G(i) > h_G(j)$. Thus h_G preserves strict inequality.*

Let n be a nonnegative integer. We shall refer to any surjective function $h: X \longrightarrow \{0, 1, 2, \ldots, n\}$ as a height function on X. If $|h^{-1}(0)| = 1$, then h is said to be a proper height function.

As an example of how the height function aids our geometric insight, we give a short proof of the well-known result that a finite nil semigroup is nilpotent.

THEOREM. *Let S be a finite nil semigroup, $|S| = n$. Then $S^n = \{0\}$.*

PROOF. S can be considered as a nil subsemigroup (via left translations) of T_X where $X = S^1$. Then $\mathrm{Fix}(S) = \{0\}$, and $h_S(x) \leqslant n - 1$ for any $x \in S$, since $|S| = n$. Thus for any $x_1, x_2, \ldots, x_n \in S$,

$$h_S(x_n) > h_S(x_{n-1}x_n) > h_S(x_{n-2}x_{n-1}x_n) > \cdots > h_S(x_1x_2 \cdots x_n)$$

by (3.9)(iii) and so $h_S(x_1x_2 \cdots x_n) = 0$ whence $x_1x_2 \cdots x_n \in \mathrm{Fix}(S) = \{0\}$.

COROLLARY. *Let S be a finite nil semigroup, $|S| = n$. The index of nilpotency is n iff S is cyclic.*

PROOF. Clearly, if S is cyclic, the index of nilpotency of S is n. Conversely, if $S^{n-1} \neq \{0\}$, there exist $x_1, x_2, \ldots, x_{n-1} \in S$ such that $x_1x_2 \cdots x_{n-1} \neq 0$.

Thus $x_2 \neq 0, x_2x_3 \neq 0, \ldots, x_2x_3 \cdots x_{n-1} \neq 0$ and no two are equal. Since $x_1, x_1x_2, \ldots, x_1x_2 \cdots x_{n-1}$ are also nonzero and no two are equal, left translation by x_1 on S^1 has rank n. But the graph of S is a chain, namely the transitive closure of

$$\begin{tikzcd} 1 \arrow[d] \\ x_{n-1} \arrow[d] \\ x_{n-2}x_{n-1} \arrow[d] \\ \vdots \arrow[d] \\ x_1x_2\cdots x_{n-1} \arrow[d] \\ 0 \end{tikzcd}$$

and thus $x_1^{n-1} \neq 0$, whence $S = \{x_1, x_1^2, \ldots, x_1^{n-1}, 0\}$.

4. **Maximal nil semigroups in T_X.** We concern ourselves now with the determination of the maximal nil semigroups, and more generally, the maximal k-nilpotent semigroups in T_X for any set X. It is sufficient by (3.4) to study nil semigroups S for which $|\mathrm{Fix}(S)| = 1$, since maximal nilpotent (and hence maximal nil) semigroups are readily seen to be graph closed.

4.1. THEOREM. *Let $\xi \in E(T_X)$ be such that $|\mathrm{Fix}(\xi)| = 1$. Then a nil semigroup N containing ξ is maximal nil iff N is graph closed and $g(N)$ is a chain with minimum element $\xi(X)$.*

PROOF. Let ξ be the zero of N and suppose that $\xi(X) = i$. Then the graph of $\{\xi\}$ is

$$\begin{tikzcd} j \arrow[dr] & k \arrow[d] & \cdots \arrow[dl] \\ & i \arrow[loop below] & \end{tikzcd}$$

where $X = \{i, j, k, \ldots\}$. This graph must be a subgraph of the graph of any nil subsemigroup of T_X containing ξ. A maximal such graph is a chain with least element i and the graph closed semigroup corresponding to such a chain is clearly a maximal nil subsemigroup of T_X.

4.2. COROLLARY. *Let X be a finite set and $\xi \in E(T_X)$, $|\mathrm{Fix}(\xi)| = 1$. Then any maximal nil subsemigroup of T_X with ξ for zero has order $(|X| - 1)!$.*

PROOF. This is immediate by the counting lemma.

It now follows from (3.4) that if $\xi \in E(T_X)$, any maximal nil subsemigroup in T_X having ξ as zero is graph closed, and its graph is a disjoint union of chains, each with minimum element.

4.3. COROLLARY. *Let $\xi \in E(T_X)$, for X finite. Then if $n_i = |X_i|$ for each $i \in \mathrm{Fix}(\xi)$, the order of any maximal nil subsemigroup of T_X having ξ as zero is* $\prod_{i \in \mathrm{Fix}(\xi)} (n_i - 1)!$.

It would certainly appear that graph closed semigroups are described completely by their graphs and one might naturally expect to be able to describe the isomorphism classes of graph closed nil semigroups in terms of graphs. By (2.11), if G is a nil graph on X, the set of all graph closed nil subsemigroups of T_X whose graphs are isomorphic to G is contained in an isomorphism class. In general, this containment is proper, i.e. there will exist graph closed nil semigroups which are isomorphic to $S(G)$, but whose graphs are not isomorphic to G. Such semigroups must of course be embedded in T_X differently, i.e. they are not isomorphic via an automorphism of T_X.

The next result describes the isomorphism classes of maximal nil subsemigroups of T_X for a finite set X.

For $n \in \mathbb{N}$, let $\pi(n)$ denote the set of all unordered partitions of n. For fixed $k \in \mathbb{N}$, $k \leqslant n$, define an equivalence relation $\overset{k}{\sim}$ on $\pi(n)$ as follows: let $p \in \pi(n)$ and let $r_k(p) = \{m \in p \mid m > k\}$. Then for $p, q \in \pi(n)$, put $p \overset{k}{\sim} q$ iff $r_k(p) = r_k(q)$.

Now, each nil semigroup $N \in S_X$ determines a partition of $|X|$, namely $p(N) = \{ |X_i| \mid i \in \mathrm{Fix}(N)\}$.

4.4. THEOREM. *Let N, M be maximal nil subsemigroups of T_X. Then $N \simeq M$ iff $p(N) \overset{2}{\sim} p(M)$.*

PROOF. Suppose that $N \simeq M$. Without loss of generality, let $\mathrm{Fix}(N) = \{1, 2, \ldots, r\}$, $\mathrm{Fix}(M) = \{1, 2, \ldots, s\}$, and $p(N) = \{n_1, n_2, \ldots, n_r\}$, $p(M) = \{m_1, m_2, \ldots, m_s\}$. We assume that the labelling is such that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ and $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$. If $n_1 > m_1$ then either $n_1 = 2$ or $n_1 > 2$. If $n_1 = 2$, then $p(N) \overset{2}{\sim} p(M)$ and we are done. Suppose $n_1 > 2$. Then N contains an element nilpotent of index $n_1 - 1$, while M has no elements of nilpotency index greater than $m_1 - 1$. This is a contradiction, since $N \simeq M$. Thus $n_1 = m_1$. Let k be the greatest integer such that $n_i = m_i$ for all $i \leqslant k$. Now if $k = \min\{r, s\}$, then $r = s$ and $p(N) = p(M)$. Suppose that $k < \min\{r, s\}$. Let $N \simeq \prod_{i=1}^r N_i$, $M \simeq \prod_{i=1}^s M_i$ as described in (3.6). Now put

$$A = \prod_{i=1}^{k} N_i, \quad B = \prod_{i=k+1}^{r} N_i, \quad C = \prod_{i=1}^{k} M_i, \quad \text{and} \quad D = \prod_{i=k+1}^{s} M_i.$$

We shall make use of the following counting functions: for any finite nil semigroup S and nonnegative integer n, let $\eta_S(n)$ be the number of elements in S of nilpotency index n, and $\mu_S(n)$ be the number of elements in S of nilpotency index less than n. Now, suppose that $n_{k+1} < m_{k+1}$, and let $m = m_{k+1} - 1$. If $m \leqslant 1$, then $p(N) \overset{2}{\sim} p(M)$. Suppose that $m > 1$. Then

$$\eta_M(m) = \eta_C(m) \cdot |D| + \mu_C(m) \cdot \eta_D(m)$$

while

$$\eta_N(m) = \eta_A(m) \cdot |B| + \mu_A(m) \cdot \eta_B(m).$$

But $\eta_B(m) = 0$, and by (2.11), $N_i \simeq M_i$ for all $i \leqslant k$, whence $A \simeq C$ and so (i) $\eta_A(m) = \eta_C(m)$, and (ii) $|A| = |C|$ which implies that $|B| = |D|$. Since $\mu_C(m)\eta_D(m) > 0$, we have $\eta_M(m) > \eta_N(m)$, a contradiction to the fact that $N \simeq M$. Thus $p(N) \overset{2}{\sim} p(M)$.

Conversely, suppose that N and M are maximal nil subsemigroups of T_X such that $p(N) \overset{2}{\sim} p(M)$. As before, suppose that $p(N) = \{n_1, n_2, \ldots, n_r\}$, $p(M) = \{m_1, m_2, \ldots, m_s\}$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$, $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$ and $N \simeq \Pi_{i=1}^r N_i$, $M \simeq \Pi_{i=1}^s M_i$. Now $n_1 \leqslant 2$ iff $m_1 \leqslant 2$ and if so, both N and M are singletons. Suppose then that $n_1, m_1 > 2$. Let k be the greatest integer such that $n_i = m_i$ for all $i \leqslant k$. Then by (2.11), there is an isomorphism $\theta_i : N_i \longrightarrow M_i$ for $i = 1, 2, \ldots, k$. Let $A = \Pi_{i=1}^k N_i$, $C = \Pi_{i=1}^k M_i$. The product map $\theta = \Pi_{i=1}^k \theta_i$ is an isomorphism of A onto C and since $N \simeq A$, $M \simeq C$ (the remaining factors in both product decompositions are zero), we have $N \simeq M$.

4.5. Corollary. *There are* $|\pi(|X|)/\overset{2}{\sim}|$ *isomorphism classes of maximal nil subsemigroups of* T_X.

We remark that if the only isomorphisms allowed are those induced by automorphisms of T_X, then there are $|\pi(|X|)|$ isomorphism classes.

Consider now the nilpotent subsemigroups of T_X. We shall describe the maximal nilpotent semigroups of given nilpotency index in terms of height functions on X.

4.6. Definition. Let $h: X \longrightarrow \{0, 1, 2, \ldots, n\}$ be a proper height function on X. Denote by $G(h)$ the directed graph on X defined as follows: for $i, j \in X$, $i \longrightarrow j \in G(h)$ iff (i) $h(i) > h(j)$; or (ii) $i = j$ and $h(i) = 0$.

4.7. Lemma. *$G(h)$ is a nil graph with $F(G) = h^{-1}(0)$ and $h = h_{G(h)}$.*

4.8. Theorem. *Let $h: X \longrightarrow \{0, 1, 2, \ldots, k\}$ be a proper height function. Then there exists a maximum k-nilpotent subsemigroup $N = N(h)$ of T_X such that $h_N = h$. Furthermore, N is graph closed and $g(N) = G(h)$.*

Proof. Let $N = S(G(h))$. Then $g(N) = G(h)$, N is graph closed and k-

nilpotent. Clearly $h_N = h$. Let $M \subset T_X$ be any nil semigroup such that $h_M = h$. We prove that $M \subset N$. Let $\gamma \in M$. Then for each $i \in X$, put $j = \gamma(i)$. Now $i \in \mathrm{Fix}(N)$ iff $h_N(i) = 0 = h(i) = h_M(i)$ iff $i \in \mathrm{Fix}(M)$, whence $j = i$. Otherwise, $i \neq j$ and so $h_M(i) > h_M(j)$, whence $h(i) > h(j)$ and so $i \longrightarrow j \in g(N)$. Thus $\gamma \in N$ and we have $M \subset N$.

4.9. COROLLARY. *For finite X, the order of $N(h)$ is computed as follows: let n_i denote the number of elements of X of height less than i, and let m_i be the number of elements of height i for $i = 1, 2, \ldots, n$. Then $|N(h)| = \Pi_{k=1}^{n} (\Sigma_{i=1}^{k} n_i)^{m_k}$.*

4.10. COROLLARY. *Let $\xi \in E(T_X)$ be such that $|\xi(X)| = 1$. Then a nil semigroup $N \subset T(X)$ with zero ξ is a maximal k-nilpotent semigroup iff $\max\{h_N(i) | i \in X\} = k$, N is graph closed, and $h_N(i) > h_N(j)$ implies that $i \longrightarrow j \in g(N)$ for all $i, j \in X$.*

4.11. COROLLARY. *Let $\{X_\alpha | \alpha \in \Lambda\}$ be a partition of X and let h_α: $X_\alpha \longrightarrow \{0, 1, 2, \ldots, m_\alpha\}$ be a proper height function for each $\alpha \in \Lambda$. If $\max\{m_\alpha | \alpha \in \Lambda\}$ exists, say it is k, then $h = \bigcup_{\alpha\in\Lambda} h_\alpha: X \longrightarrow \{0, 1, 2, \ldots, k\}$ is a height function on X and there exists a maximum k-nilpotent semigroup $N \subset T_X$ such that $h_N = h$.*

4.12. COROLLARY. *Let $N \in S_X$ be k-nilpotent. Then there exists a maximum k-nilpotent subsemigroup M of T_X such that $N \subset M$ and $h_M = h_N$.*

In particular, for finite X, each nil $N \in S_X$ is contained in a maximum nil $M \in S_X$ such that $h_N = h_M$.

5. Ideal chains of nil semigroups.

5.1. THEOREM. *Let $N, M \in S_X$ be maximal nilpotent of index n and m respectively. If $N \subset M$ then $N \,\Delta\, M$.*

PROOF. Let $\alpha \in N$, $\beta \in M\backslash N$. We must show that $\alpha\beta$ and $\beta\alpha$ belong to N. Since N is maximal n-nilpotent, this will be done if we show that for all $i \in X\backslash \mathrm{Fix}(N)$, (i) $h_N(i) > h_N \cdot \alpha\beta(i)$, and (ii) $h_N(i) > h_N \cdot \beta\alpha(i)$. The proofs of (i) and (ii) are similar, and we just prove (i). Let $i \in X\backslash\mathrm{Fix}(N)$. Note that $\mathrm{Fix}(M) = \mathrm{Fix}(N)$. Put $k = \beta(i)$, $j = \alpha(k)$. We want to show that $h_N(i) > h_N(j)$. If $h_N(i) < h_N(k)$, then there exists $\gamma \in N$ such that $\gamma(k) = i$. But then $\gamma \in M$ and so $\gamma\beta(i) = i$, whence $i \in \mathrm{Fix}(M)$, a contradiction. Thus $h_N(i) \geqslant h_N(k)$. If $k \in \mathrm{Fix}(N)$, then $k = j$ and $h_N(i) > 0 = h_N(j)$. If $k \notin \mathrm{Fix}(N)$, then $h_N(k) > h_N(j)$ and so $h_N(i) > h_N(j)$.

For finite X, every graph closed subsemigroup of any maximal nil subsemigroup N of T_X is a meta-ideal of N. We need some preliminary results to establish this, beginning with a result for arbitrary X.

5.2. LEMMA. *Let $N \in S_X$ be nil and let $i, j \in \mathrm{Fix}(N)$, $i \neq j$. Then for any $x \in X_i$, $y \in X_j$, $x \longrightarrow y \notin \mathcal{I} \cdot g(N)$.*

PROOF. Suppose that for some $x \in X_i$, $y \in X_j$, $x \longrightarrow y \in \mathcal{I} \cdot g(N)$. Then since $y \longrightarrow j \in g(N)$, we must have $x \longrightarrow j \in g(N)$, and so $x \in X_j$, a contradiction.

Let N_i, $i \in \mathrm{Fix}(N)$, be the subsemigroup of N as described in (2.11).

5.3. COROLLARY. $\mathcal{I} \cdot g(N) = \bigcup_{i \in \mathrm{Fix}(N)} \mathcal{I} \cdot g(N_i)$.

5.4. THEOREM. *Let $N \in S_X$, X finite, be maximal nil and let S be a graph closed subsemigroup of N. If $\mathcal{I} \cdot g(S) \cap g(S) \cap g(S)$, then $S = N$.*

PROOF. By (5.3), we may assume that $\mathrm{Fix}(S) = \mathrm{Fix}(N)$ is a singleton. Suppose now that $S \neq N$, whence there exists an integer n for which $|h_S^{-1}(n)| > 1$. Let m be the smallest such integer. Then $m > 0$ by assumption, and the elements of X of height less than m form a chain under $\leqslant_S$, with a maximum element, say k. Let $i, j \in h_S^{-1}(m)$, whence $i \longrightarrow k$ and $j \longrightarrow k$ belong to $g(S)$. Now exactly one of $i \longrightarrow j$ or $j \longrightarrow i$ belongs to $g(N)$. Suppose that $i \longrightarrow j$ belongs to $g(N)$. We show that $i \longrightarrow j \in g(S)$ and hence $h_S(i) > h_S(j)$, a contradiction. Note that since $i \longrightarrow j \in g(N)$, $j \longrightarrow i \notin g(S)$. Now let l be any element of $X_i(S)$ of maximum height. Since $j \longrightarrow l \in g(N)$ implies $j \longrightarrow i \in g(N)$, we have $j \longrightarrow l \notin g(N)$. Thus $l \longrightarrow j \in g(N)$ and since l is maximal in $X_i(S)$ under $\leqslant_S$, $l \longrightarrow j \in \mathcal{I} \cdot g(S)$. Thus $l \longrightarrow j \in g(S)$. Now repeat this procedure for all elements of $X_i(S)$ of height one less than the maximum height in $X_i(S)$. We obtain that for all elements l in $X_i(S)$ at this height, $l \longrightarrow j \in g(S)$. By induction, we obtain $i \longrightarrow j \in g(S)$, the contradiction we are seeking. Thus no two elements of X have the same height relative to S and so S is maximal nil, whence $S = N$.

5.5. COROLLARY. *Let X be finite. Every graph closed subsemigroup of a maximal nil subsemigroup N of $\mathcal{T}_X$ is a meta-ideal of N.*

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